FOURIER SERIES

1.1 Introduction

Fourier series introduced by a French physicist Joseph Fourier (1768-1830), is a mathematical tool that converts some specific periodic signals into everlasting sinusoidal waveforms, which is of utmost importance in scientific and engineering applications.

The Fourier series allows us to model any arbitrary periodic signal or function $f(x)$ in the form $\frac{a_0}{2} + (a_1 cosx + a_2 cos2x + \cdots) + (b_1 sinx + b_2 sin2x + \cdots)$ the interval $[C, C + 2l]$ under some conditions called **Dirichlet's conditions** as given below:

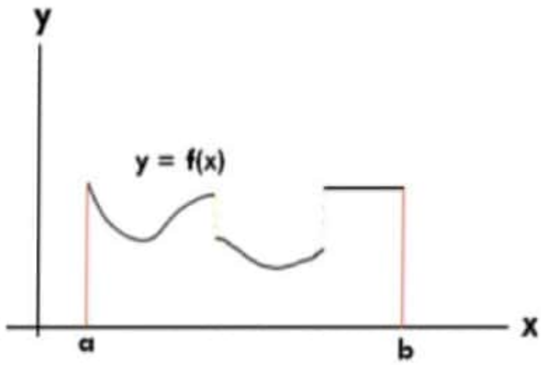
(i) $f(x)$ is periodic with a period $2l$

(ii) $f(x)$ and its integrals are finite and single valued in $[C, C + 2l]$

(iii) $f(x)$ is piecewise continuous* in the interval $[C, C + 2l]$

(iv) $f(x)$ has a finite no of maxima & minima in $[C, C + 2l]$

* A function $f(x)$ is said to be **piecewise continuous** in an interval $[a, b]$, if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite left and right hand limits i.e. it is bounded. In other words, a piecewise continuous function is a function that has a finite number of discontinuities and doesn't blow up to infinity anywhere in the given interval.



Thus any function $f(x)$ defined in $[C_1, C_2]$ and satisfying Dirichlet's conditions can be expressed in Fourier series given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$ where a_0, a_n b_n are constants called the Fourier coefficients of $f(x)$, required to determine any function into Fourier series.

Periodic Functions

A function $f(x)$ is said to be periodic if there exists a positive number T such that $f(x + T) = f(x) \;\forall\; x \in R$.

Here T is the smallest positive real number such that $f(x + T) = f(x) \;\forall\; x \in R$ and is called the fundamental period of $f(x)$.

We know that $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic functions with period 2π whereas $\tan x$ and $\cot x$ are periodic with a period π. The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$, while fundamental period of $\tan nx$ is $\frac{\pi}{n}$.

Example1 Determine the period of the following functions:

i. $\sin(5x+3)$ ii. $\sin^2 x$ iii. $|\cos x|$ iv. $\sin x + \cos x$

v. K , where K is a constant vi. $\cos x + \frac{1}{3}\cos 2x + \frac{1}{2}\cos\frac{x}{3}$

Solution: i. $f(x) = \sin(5x+3)$

$\sin 5x$ is a periodic function with a period $\frac{2\pi}{5}$

$\therefore f(x)$ is periodic with a period $\frac{2\pi}{5}$

ii. $f(x) = \sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$

$\cos 2x$ is a periodic function with a period $\frac{2\pi}{2} = \pi$

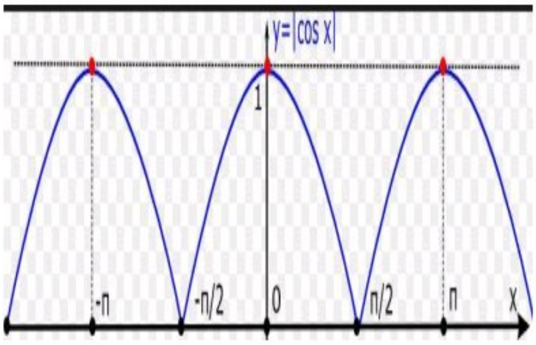
$\therefore f(x)$ is periodic with a period π

iii. $f(x) = |\cos x| = \sqrt{\cos^2 x} \quad \because |x| = \sqrt{x^2}$

$= \sqrt{\frac{1+\cos 2x}{2}}$

Now $\cos 2x$ is a periodic function with a period $\frac{2\pi}{2} = \pi$

$\therefore f(x)$ is periodic with a period π



Note: From the graph of $|\cos x|$, period is π

iv. $f(x) = (\sin x + \cos x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$

$= \sqrt{2}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

Now $\sin x$ is a periodic function with a period 2π

$\therefore f(x)$ is periodic with a period 2π

Or

Periods of $\sin x$ is 2π and period of $\cos x$ is also 2π

$\therefore f(x)$ is periodic with a period which is lowest common multiple (l.c.m) of $(2\pi, 2\pi) = 2\pi$

v. $f(x) = K$, where K is a constant

$\Rightarrow f(x+T) = K = f(x)\ \forall\ x \in R$

$\therefore f(x)$ is periodic function, but fundamental period of $f(x)$ i.e. T can not be defined since it is the smallest positive real number.

vi. $f(x) = \cos x + \frac{1}{3}\cos 2x + \frac{1}{2}\cos\frac{x}{3}$

Periods of $\cos x$, $\cos 2x$ and $\cos\frac{x}{3}$ are 2π, π and 6π respectively.

$\therefore f(x)$ is periodic with a period which is lowest common multiple i.e.

(l.c.m) of $(2\pi, \pi, 6\pi) = 6\pi$

Example 2 Determine a sinusoidal periodic function whose period is given by:

i. $\frac{\pi}{k}$ ii. $\frac{k}{2}$ iii. $2 - k$

Solution: The function $\sin nx$ is periodic with period $\frac{2\pi}{n}$

i. $\frac{\pi}{k} = \frac{2\pi}{n} \Rightarrow n = 2\text{k}$

$\therefore$ The required periodic function is $\sin 2kx$

ii. $\frac{k}{2} = \frac{2\pi}{n} \Rightarrow n = \frac{4\pi}{k}$

$\therefore$ The required periodic function is $\sin\frac{4\pi x}{k}$

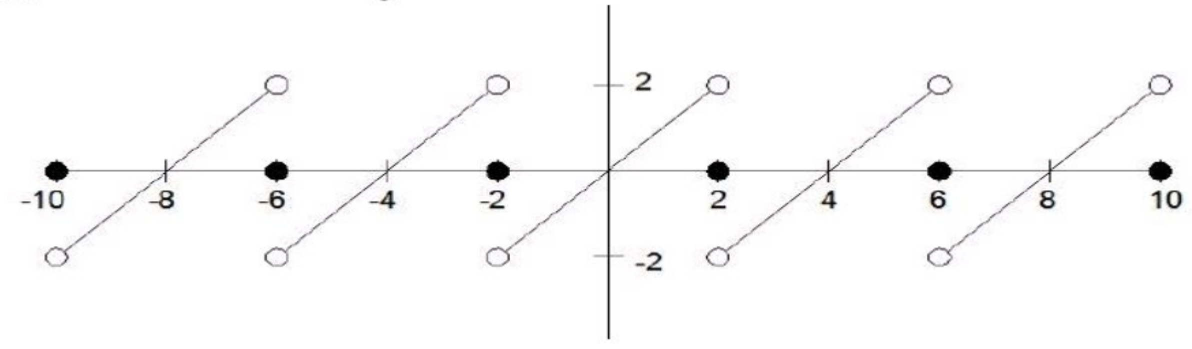
iii. $2 - k = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{2-k}$

$\therefore$ The required periodic function is $\sin\frac{2\pi x}{2-k}$

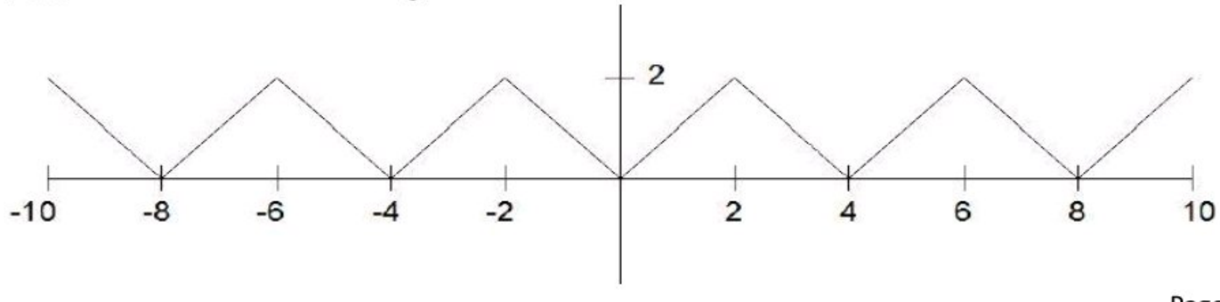
Example 3 Draw the graph of the following functions which are assumed to be periodic with a period '4', such that $f(x + 4) = f(x)$

i. $f(x) = x,\ -2 < x < 2$ ii. $f(x) = |x|,\ -2 < x < 2$

Solution: i. Potting graph of $f(x) = x$ in the interval $(-2,2)$ and repeating periodically in the intervals $(-6,-2)$, $(-10,-6)$ on the left and $(2,6)$, $(6,10)$ on the right, since period of $f(x)$ is given to be '4' units



ii. Potting graph of $f(x) = |x|$ in the interval $(-2,2)$ and repeating periodically in the intervals $(-6,-2)$, $(-10,-6)$ on the left and $(2,6)$, $(6,10)$ on the right, since period of $f(x)$ is given to be '4' units



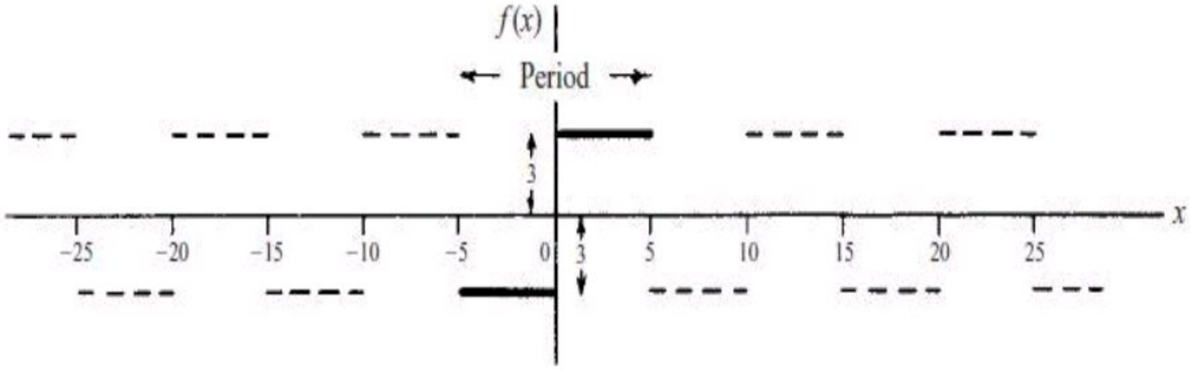
Example 4 Draw the graph of the following functions which are assumed to be periodic with a period 'T', such that $f(x+T) = f(x)$

i. $f(x) = \begin{cases} -3, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$ $\quad f(x+10) = f(x)$

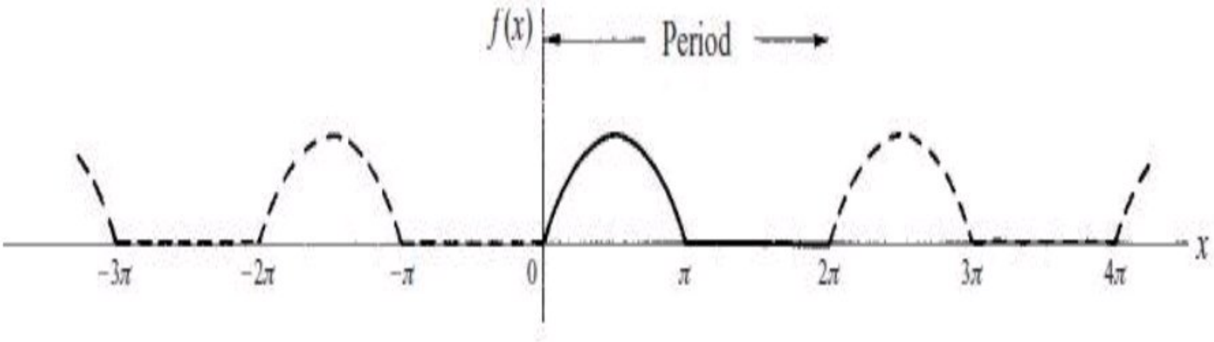
ii. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$ $\quad f(x+2\pi) = f(x)$

iii. $f(x) = \begin{cases} 0, & 0 \le x < 2 \\ 1, & 2 \le x < 4 \\ 0, & 4 \le x < 6 \end{cases}$ $\quad f(x+6) = f(x)$

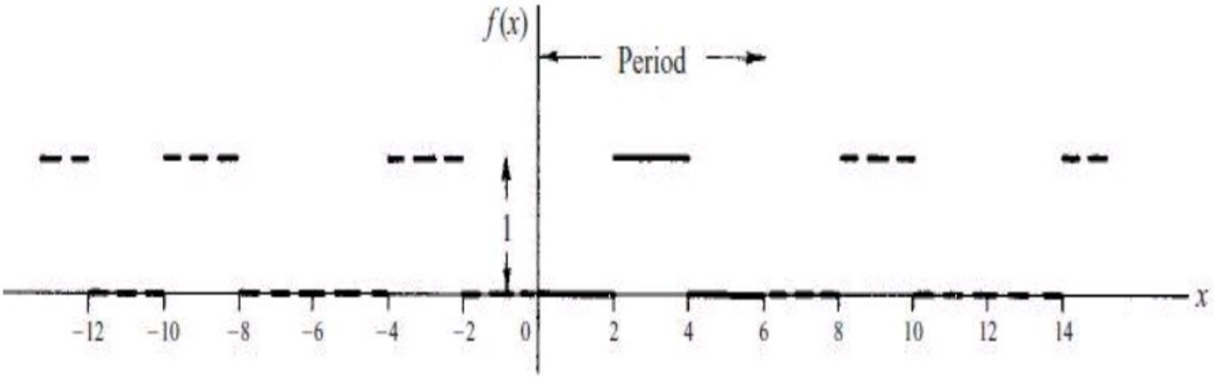
Solution: i. Plotting graph of $f(x)$ in the interval $(-5,5)$ and repeating periodically taking period of 10 units



ii. Plotting graph of $f(x)$ in the interval $(0,2\pi)$ and repeating periodically taking period of 2π units



iii. Plotting graph of $f(x)$ in the interval $(0,6)$ and repeating periodically taking period of 6 units



Some useful results in computation of the Fourier series:

If m, n are non - zero integers then:

(i) $\int_c^{c+2\pi} sinnx\, dx = -\left[\frac{cosnx}{n}\right]_c^{c+2\pi} = 0$

(ii) $\int_c^{c+2\pi} cosnx\, dx = 0, n \neq 0$

(iii) $\int_c^{c+2\pi} sinmx.\, sinnx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

(iv) $\int_c^{c+2\pi} cosmx.\, cosnx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

(v) $\int_c^{c+2\pi} sinmx.\, cosnx\, dx = 0$

(vi) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

(vii) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

(viii) $\sin n\pi = 0$

(ix) $\cos n\pi = (-1)^n$

(x) Integration by parts when first function vanishes after a finite number of differentiations:

If u and v are functions of x

$$\int \boldsymbol{u.v\, dx = uv_1 - u^{(1)}v_2 + u^{(2)}v_3 - u^{(3)}v_4} + \cdots$$

Here $\boldsymbol{u^{(n)}}$is derivative of $\boldsymbol{u^{(n-1)}}$ and $\boldsymbol{v_n}$ is integral of $\boldsymbol{v_{n-1}}$

For example

$$\int \boldsymbol{x^2. \sin nx\ dx} = (\boldsymbol{x^2})\left(-\frac{\boldsymbol{\cos nx}}{\boldsymbol{n}}\right) - (\boldsymbol{2x})\left(-\frac{\boldsymbol{sinnx}}{\boldsymbol{n^2}}\right) + (\boldsymbol{2})\left(\frac{\boldsymbol{cosnx}}{\boldsymbol{n^3}}\right)$$

$$= -\boldsymbol{x^2 \cos x\ + 2x \sin x + 2 \cos x}$$

$$= -\frac{\boldsymbol{x^2 \cos nx}}{\boldsymbol{n}} + \frac{\boldsymbol{2x\, sinnx}}{\boldsymbol{n^2}} + \frac{\boldsymbol{2cosnx}}{\boldsymbol{n^3}}$$

1.2 Euler's Formulae (To find Fourier coefficients a_0, a_n, b_n when interval length is 2π)

If a function $f(x)$ satisfies Dirichlet's conditions, it can be expanded into Fourier series given by $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$ …… ①

To find a_0

Integrating both sides of ① with respect to x within the limits C to C+2π

$$\int_C^{C+2\pi} f(x)dx = \frac{a_0}{2}\int_C^{C+2\pi} dx + \int_C^{C+2\pi}(a_1cosx + a_2cos2x + \cdots)dx +$$
$$\int_C^{C+2\pi}(b_1sinx + b_2sin2x + \cdots)dx$$

$\Rightarrow \int_C^{C+2\pi} f(x)dx = \frac{a_0}{2}[x]_C^{C+2\pi} + 0 + 0$ using results (i) and (ii)

$$= \frac{a_0}{2}[C + 2\pi - C] = a_0\pi$$

$$\Rightarrow\ a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx$$

To find a_n

Multiplying both sides of ① by $cosnx$ and integrating respect to x within the limits C to C+2π

$$\int_C^{C+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_C^{C+2\pi}\cos nx\ dx +$$

$$\int_C^{C+2\pi}(a_1 cosx\ \cos nx + \cdots a_n\cos nx\ \cos nx + \cdots)dx +$$

$$\int_C^{C+2\pi}(b_1 sinx\ \cos nx\ + \cdots + b_n\sin nx\ \cos nx + \cdots)dx$$

$$\Rightarrow \int_C^{C+2\pi} f(x)\cos nx\ dx = 0 + a_n\int_C^{C+2\pi} cos^2 nx\,dx + 0$$ using results (i),(ii),(iv), (v)

$$= \frac{a_n}{2}\int_C^{C+2\pi}(1+\cos 2nx)\,dx$$

$$= \frac{a_n}{2}\left[x + \frac{\sin 2nx}{2n}\right]_C^{C+2\pi}$$

$$= \frac{a_n}{2}[C + 2\pi - C] = a_n\pi$$

$$\Rightarrow\ a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx$$

To find b_n

Multiplying both sides of ① by $\sin nx$ and integrating respect to x within the limits C to C+2π

$$\int_C^{C+2\pi} f(x)\ \sin nx\,dx = \frac{a_0}{2}\int_C^{C+2\pi}\sin nx\ dx +$$

$$\int_C^{C+2\pi}(a_1 cosx\ \sin nx + \cdots a_n cosnx\ \ \sin nx + \cdots)dx +$$

$$\int_C^{C+2\pi}(b_1\sin x\ \sin nx\ + \cdots + b_n\sin nx\ \sin nx + \cdots)dx$$

$$\Rightarrow \int_C^{C+2\pi} f(x)\sin nx\,dx = 0 + 0 + b_n\int_C^{C+2\pi} sin^2 nx\,dx$$ using results (i),(ii),(iii), (v)

$$= \frac{b_n}{2}\int_C^{C+2\pi}(1-\cos 2nx)\,dx$$

$$= \frac{b_n}{2}\left[x - \frac{\sin 2nx}{2n}\right]_C^{C+2\pi}$$

$$= \frac{b_n}{2}[C + 2\pi - C] = b_n\pi$$

$$\Rightarrow\ b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

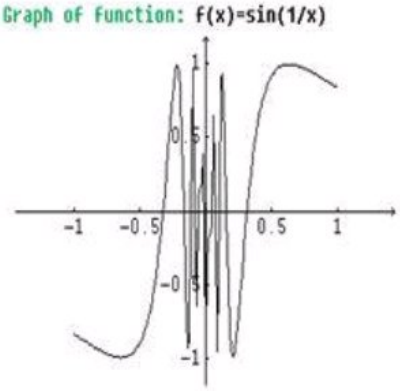
Example 5 State giving reasons whether the following functions can be expanded into Fourier series in the interval $[-\pi, \pi]$

i. $\sin\frac{1}{x}$ ii. $\operatorname{cosec} x$ iii. $\frac{1}{2-x}$

Solution: i. $f(x) = \sin\frac{1}{x}$ is not single valued at $x = 0$.

Also graph of $\sin\frac{1}{x}$ oscillates infinite number of times in vicinity of $x = 0$ as shown in adjoining figure. $\therefore f(x)$ is having infinite number of maxima and minima in $[-\pi, \pi]$.

Hence $f(x) = \sin\frac{1}{x}$ violates Dirichlet's conditions and cannot be expanded into Fourier series.



ii. $f(x) = \operatorname{cosec} x$ is not piecewise continuous in the interval $[-\pi, \pi]$ as

$$\lim_{x\to 0^-} f(x) = -\infty \text{ and } \lim_{x\to 0^+} f(x) = +\infty$$

Hence $f(x) = \operatorname{cosec} x$ violates Dirichlet's conditions and cannot be expanded into Fourier series.

iii. $f(x) = \frac{1}{2-x}$ is not piecewise continuous in the interval $[-\pi, \pi]$ as

$$\lim_{x\to 2^-} f(x) = +\infty \text{ and } \lim_{x\to 2^+} f(x) = -\infty$$

Hence $f(x) = \frac{1}{2-x}$ violates Dirichlet's conditions and cannot be expanded into Fourier series.

Example 6 If $f(x + 2\pi) = f(x)$, find the Fourier expansion $f(x) = x$ in the interval $[0, 2\pi]$

Hence or otherwise prove that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Solution: $f(x) = x$ is integrable and piecewise continuous in the interval $[0, 2\pi]$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \quad \ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\, dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{sinnx}{n}\right) - (1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x\ \frac{sinnx}{n} + \frac{cosnx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{cosnx}{n^2}\right]_0^{2\pi} \qquad \because sinnx = 0 \text{ when } x = 0 \text{ or } x = 2\pi$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0 \qquad \because \cos 2n\pi = 1$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= -\frac{1}{\pi}\left[\frac{2\pi}{n}\right] = -\frac{2}{n} \qquad \because sinnx = 0 \text{ when } x = 0 \text{ or } x = 2\pi \text{ and } \cos 2n\pi = 1$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \pi - 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

Putting $x = \frac{\pi}{2}$ on both sides

$$\frac{\pi}{2} = \pi - 2\left[\frac{1}{1} + 0 - \frac{1}{3} + 0 + \frac{1}{5} + \cdots\right]$$

$$\Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Example 7 If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of

$$f(x) = \begin{cases} 0, -\pi \le x \le 0 \\ x, \ 0 \le x \le \pi \end{cases}$$

Hence or otherwise prove that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Solution: $f(x)$ is integrable and piecewise continuous in the interval $[-\pi, \pi]$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,\mathrm{dx} + \int_0^{\pi} x dx\right] = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ \cos nx\,\mathrm{dx} + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{sinnx}{n}\right) - (1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x\,sinnx}{n} + \frac{cosnx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \because sinnx = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1] = \begin{cases} \frac{-2}{\pi n^2} & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ \sin nx\,\mathrm{d}x + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\pi(-1)^n}{n}\right] \qquad \because \frac{sinnx}{n^2} = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= -\frac{1}{n}[(-1)^n] = \frac{(-1)^{n+1}}{n} = \begin{cases} \frac{1}{n} & , n\ is\ odd \\ -\frac{1}{n} & , n\ is\ even \end{cases}$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \cdots\right]$$

Putting $x = \frac{\pi}{2}$ on both sides

$$\frac{\pi}{2} = \frac{\pi}{4} - 0 + \left[\frac{1}{1} - 0 - \frac{1}{3} - 0 + \frac{1}{5} - \cdots\right]$$

$$\Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Example 8 If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of

$$f(x) = \begin{cases} \cos x, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$$

Solution: $f(x)$ is integrable and piecewise continuous in the interval $[-\pi, \pi]$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ \ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x\ \mathrm{d}x + \int_0^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{\pi}[\sin x]_{-\pi}^{0} - \frac{1}{\pi}[\cos x]_0^{\pi}$$

$$= 0 - \frac{1}{\pi}[\cos\pi - \cos 0] = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x\ \cos nx\,\mathrm{d}x + \int_0^{\pi} \sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} [\cos(n+1)x + \cos(n-1)x]\mathrm{d}x + \int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_{0}^{\pi}, n \neq 1$$

$$= 0 + \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \frac{1}{2\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], n \text{ is } odd (n \neq 1) \\ \frac{1}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], \quad n \text{ is } even \end{cases} = \begin{cases} 0, \quad n \text{ is } odd (n \neq 1) \\ \frac{-2}{\pi(n^2-1)}, \quad n \text{ is } even \end{cases}$$

$$a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x \cos x \, \mathrm{d}x + \int_{0}^{\pi} \sin x \cos x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos^2 x \, \mathrm{d}x + \frac{1}{2}\int_{0}^{\pi} \sin 2x \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\int_{-\pi}^{0} (1 + \cos 2x) \, \mathrm{d}x + \frac{1}{2}\int_{0}^{\pi} \sin 2x \, dx\right]$$

$$= \frac{1}{2\pi}\left[x + \frac{\sin 2x}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[-\frac{\cos 2x}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}[\pi] - \frac{1}{4\pi}[1 - 1] = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_{C}^{C+2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x \, \sin nx \, \mathrm{d}x + \int_{0}^{\pi} \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} [\sin(n+1)x + \sin(n-1)x] \mathrm{d}x + \int_{0}^{\pi} [\cos(n-1)x - \cos(n+1)x] dx\right]$$

$$= -\frac{1}{2\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_{0}^{\pi}, n \neq 1$$

$$= -\frac{1}{2\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} - \frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}\right] + 0 \quad \because sinnx = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= -\frac{1}{2\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} - \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \begin{cases} -\frac{1}{2\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} - \frac{1}{n+1} - \frac{1}{n-1}\right], n \text{ is } odd (n \neq 1) \\ -\frac{1}{2\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right], \quad n \text{ is } even \end{cases} = \begin{cases} 0, \quad n \text{ is } odd (n \neq 1) \\ \frac{-2n}{\pi(n^2-1)}, \quad n \text{ is } even \end{cases}$$

$$b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos x \sin x \, \mathrm{d}x + \int_{0}^{\pi} \sin x \sin x \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\int_{-\pi}^{0} \sin 2x \, \mathrm{d}x + \int_{0}^{\pi} \sin^2 x \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\int_{-\pi}^{0} \sin 2x \, \mathrm{d}x + \frac{1}{2}\int_{0}^{\pi} (1 - \cos 2x) \, dx\right]$$

$$= -\frac{1}{2\pi}\left[\frac{\cos 2x}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_{0}^{\pi}$$

$$= -\frac{1}{4\pi}[1 - 1] + \frac{1}{2\pi}[\pi - 0 - 0 + 0] = \frac{1}{2}$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \cdots\right] + \frac{1}{2} - \frac{2}{\pi}\left[\frac{2\sin 2x}{2^2-1} + \frac{4\sin 4x}{4^2-1} + \cdots\right]$$

$$\Rightarrow f(x) \approx \frac{1}{\pi} + 1 - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \cdots\right] - \frac{2}{\pi}\left[\frac{2\sin 2x}{2^2-1} + \frac{4\sin 4x}{4^2-1} + \cdots\right]$$

Example 9 If $f(x+2\pi) = f(x)$, find the Fourier expansion $f(x) = e^{ax}$ in the interval $[-\pi, \pi]$

Solution: $f(x) = e^{ax}$ is integrable and piecewise continuous in the interval $[-\pi, \pi]$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ \ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx \ = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax} dx$$

$$= \frac{1}{a\pi}[e^{ax}]_{-\pi}^{\pi} = \frac{1}{a\pi}[e^{a\pi} - e^{-a\pi}] = \frac{2}{a\pi}\sinh a\pi \quad \because \frac{e^x - e^{-x}}{2} = \sinh x$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\, dx$$

$$= \frac{1}{\pi(a^2+n^2)}[e^{ax}\ (a\cos nx + n\sin nx)]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}[e^{a\pi}\ (a\cos n\pi + n\sin n\pi) - e^{-a\pi}\ (a\cos n\pi - n\sin n\pi)]$$

$$= \frac{a(-1)^n}{\pi(a^2+n^2)}[e^{a\pi} - e^{-a\pi}] = \frac{2a(-1)^n}{\pi(a^2+n^2)}\sinh a\pi$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\, dx$$

$$= \frac{1}{\pi(a^2+n^2)}[e^{ax}\ (a\sin nx - n\cos nx)]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}[e^{a\pi}\ (a\sin n\pi - n\cos n\pi) - e^{-a\pi}\ (a\sin n\pi - n\cos n\pi)]$$

$$= \frac{-n(-1)^n}{\pi(a^2+n^2)}[e^{a\pi} - e^{-a\pi}] = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)}\sinh a\pi$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \frac{\sinh a\pi}{\pi}\left[\frac{1}{a} + 2a\left[-\frac{\cos x}{(a^2+1^2)} + \frac{\cos 2x}{(a^2+2^2)} - \frac{\cos 3x}{(a^2+3^2)} + \cdots\right] + 2\left[\frac{\sin x}{(a^2+1^2)} - \frac{2\sin 2x}{(a^2+2^2)} + \right.\right.$$

3sin3*xa*2+32−...

1.2.1 Determination of Function Values at the Points of Discontinuity

A function satisfying Dirichlet's conditions may be expanded into Fourier series if it is discontinuous at a finite number of points.

Let the function be defined in (a, b) as

$$f(x) = \begin{cases} f_1(x), & a < x < x_o \\ f_2(x), & x_o < x < b \end{cases}$$

1. **To find $f(x)$ at $x = a$ or $x = b$ (End points discontinuity)**
   Since $f(a)$ and $f(b)$ are not defined in the interval (a, b)
   $\therefore f(a) = f(b) = \frac{1}{2}[(RHL\ at\ x = a) + (LHL\ at\ x = b)]$
   $= \frac{1}{2}\left[\lim_{x\to a^+} f(x) + \lim_{x\to b^-} f(x)\right]$
2. **To find $f(x)$ at $x = x_o$ (Mid point discontinuity)**
   Since $f(x_o)$ is not defined in the interval (a, b)
   $\therefore f(x_o) = \frac{1}{2}[(LHL\ at\ x = x_o) + (RHL\ at\ x = x_o)]$
   $= \frac{1}{2}\left[\lim_{x\to x_o^-} f(x) + \lim_{x\to x_o^+} f(x)\right]$

Example 10 If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of

$$f(x) = \begin{cases} -\pi, -\pi < x < 0 \\ x,\ 0 < x < \pi \end{cases}$$

Hence or otherwise prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

Solution: $f(x)$ is integrable and piecewise continuous in the interval $(-\pi, \pi)$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ \ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx \quad = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\ \mathrm{d}x + \int_0^{\pi} xdx\right] = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\ \cos nx\ \mathrm{d}x + \int_0^{\pi} x\cos nx\, dx\right]$$

$$= \frac{-\pi}{\pi}\left[\frac{sinnx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x)\left(\frac{sinnx}{n}\right) - (1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= 0 + \frac{1}{\pi}\left[\frac{x\ sinnx}{n} + \frac{cosnx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \because sinnx = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1] = \begin{cases} \frac{-2}{\pi n^2} & , n\ is\ odd \\ 0 & , n\ is\ even \end{cases}$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\ \sin nx\ \mathrm{d}x + \int_0^{\pi} x\sin nx\, dx\right]$$

$$= \frac{\pi}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \left[\frac{1}{n} - \frac{(-1)^n}{n}\right] - \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n}\right] \qquad \because \frac{sinnx}{n^2} = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= \frac{1}{n}[1 - 2(-1)^n] = \begin{cases} \frac{3}{n} & , n \ is \ odd \\ -\frac{1}{n} & , n \ is \ even \end{cases}$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left[\frac{3\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

Putting $x = 0$ on both sides

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] + 0 \ldots ②$$

Since $f(0)$ is not defined in the interval $(-\pi, \pi)$

$$\therefore f(0) = \frac{1}{2}[(LHL \ at \ x = 0) + (RHL \ at \ x = 0)]$$

$$= \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0} f(0 - h) + \lim_{h\to 0} f(0 + h)\right]$$

$$= \frac{1}{2}[-\pi + 0] = -\frac{\pi}{2} \ \ldots ③$$

Using ③in ②, we get

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

Example 11 If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of $f(x) = x + x^2$ in the interval $(-\pi, \pi)$

Hence or otherwise prove that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}$

Solution: $f(x) = x + x^2$ is integrable and piecewise continuous in the interval $(-\pi, \pi)$

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ \ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\ \cos nx\, dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{sinnx}{n}\right) - (1+2x)\left(\frac{-cosnx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\left[(1+2x)\, cosnx\right]_{-\pi}^{\pi} \quad \because sinnx = 0 \text{ when } x = -\pi \text{ or } x = \pi$$

$$= \frac{1}{\pi n^2}\left[(1+2\pi)\, cosn\pi - (1-2\pi)\, cosn\pi\right] \quad \because \cos(-n\pi) = cosn\pi$$

$$= \frac{1}{\pi n^2}\left[4\pi\, cosn\pi\right] = \frac{4(-1)^n}{n^2} = \begin{cases} \frac{-4}{n^2} & , n \text{ is odd} \\ \frac{4}{n^2} & , n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\ \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi} \quad \because sinnx = 0 \text{ when } x = -\pi \text{ or } x = \pi$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\left(\frac{-\cos n\pi}{n}\right) + \left(\frac{2\cos n\pi}{n^3}\right) - (-\pi+\pi^2)\left(\frac{-\cos n\pi}{n}\right) - \left(\frac{2\cos n\pi}{n^3}\right)\right]$$

$$\because \cos(-n\pi) = \cos n\pi$$

$$= \frac{-1}{\pi n}\left[2\pi\, cosn\pi\right] = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n} = \begin{cases} \frac{2}{n} & , n \text{ is odd} \\ \frac{-2}{n} & , n \text{ is even} \end{cases}$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

Putting $x = \pi$ on both sides

$$f(\pi) \approx \frac{\pi^2}{3} + 4\left[-\frac{\cos \pi}{1^2} + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} + \cdots\right] + 0 \quad \ldots ②$$

Now since $f(\pi)$ is not defined in the interval $(-\pi, \pi)$

$$\therefore f(\pi) = \frac{1}{2}\left[(LHL\ at\ x = \pi) + (RHL\ at\ x = -\pi)\right]$$

$$= \frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to-\pi^+} f(x)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0} f(\pi - h) + \lim_{h\to 0} f(-\pi + h)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0}\{(\pi - h) + (\pi - h)^2 + (-\pi + h) + (-\pi + h)^2\}\right]$$

$$\Rightarrow f(\pi) = \frac{1}{2}\left[\pi + \pi^2 - \pi + \pi^2\right] = \pi^2 \quad \ldots ③$$

Using ③in ②, we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}$$

1.3 Fourier Series for Arbitrary Period Length

Let $f(x)$ be a periodic function defined in the interval $[C, C+2l]$, then

$$f(x) \approx \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \;\ldots\ldots ①$$

$$a_0=\frac{1}{l}\int_c^{c+2l} f(x)dx$$

$$a_n=\frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n=\frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$$

Note: If the interval length is 2π, putting $2l=2\pi$ i.e. $=\pi$, then ① may be rewritten as $f(x) \approx \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n cosnx+\sum_{n=1}^{\infty} b_n sinnx$, which is Fourier series expansion in the interval $[C, C+2\pi]$.

Also $a_0=\frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

$$a_n=\frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Example 12 If $f(x+10)=f(x)$, find the Fourier series expansion of the function

$$f(x)=\begin{cases}0, & -5\le x\le 0\\ 3, & 0\le x\le 5\end{cases}$$

Solution: Let $f(x) \approx \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here interval is $[-5,5]$, $\therefore 2l=10 \Rightarrow l=5$

Putting $l=5$ in ①

$$f(x) \approx \frac{a_0}{2}+\sum a_n \cos\frac{n\pi x}{5}+\sum b_n \sin\frac{n\pi x}{5} \;\ldots\ldots ①$$

$$a_0=\frac{1}{l}\int_c^{c+2l} f(x)dx=\frac{1}{5}\int_{-5}^{5} f(x)dx=\frac{1}{5}\int_{-5}^{0} 0dx+\frac{1}{5}\int_0^5 3dx=\frac{3}{5}[x]_0^5=3$$

$$a_n=\frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}dx$$

$$=\frac{1}{5}\int_{-5}^{0} 0\cos\frac{n\pi x}{l}dx+\frac{1}{5}\int_0^5 3\cos\frac{n\pi x}{5}dx=0+\frac{3}{5}\left[\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right]_0^5=0$$

$$b_n=\frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}dx$$

$$= \frac{1}{5}\int_{-5}^{0} 0 \ \sin\frac{n\pi x}{l}dx + \frac{1}{5}\int_{0}^{5} 3\sin\frac{n\pi x}{5}dx$$

$$= 0 - \frac{3}{5}\left[\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right]_0^5 = -\frac{3}{n\pi}[\cos n\pi - \cos 0]$$

$$= -\frac{3}{n\pi}[(-1)^n - 1] = \begin{cases} \frac{6}{n\pi} \ , n \ is \ odd \\ 0 \ , n \ is \ even \end{cases}$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \frac{3}{2} + \frac{6}{\pi}\left[\frac{\sin\frac{\pi x}{5}}{1} + \frac{\sin\frac{3\pi x}{5}}{3} + \frac{\sin\frac{5\pi x}{5}}{5} + \cdots\right]$$

$$\Rightarrow f(x) \approx \frac{3}{2} + \frac{6}{\pi}\left[\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin \pi x + \cdots\right]$$

Example 13 If $f(x+2) = f(x)$, find the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$$

Solution: Let $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here interval is $[0,2]$, $\therefore 2l = 2 \Rightarrow l = 1$

Putting $l = 1$ in ①

$f(x) \approx \frac{a_0}{2} + \sum a_n cosn\pi x + \sum b_n sinn\pi x$ …… ①

Now $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx \ = \int_0^2 f(x)dx = \int_0^1 \pi x dx + \int_1^2 \pi(2-x)dx$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \frac{\pi}{2} + \pi\left[4 - 2 - 2 + \frac{1}{2}\right] = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) \ \cos\frac{n\pi x}{l}dx = \int_0^2 f(x) \ \cos n\pi x \, dx$$

$$= \int_0^1 \pi x \cos n\pi x \, dx + \int_1^2 \pi(2-x)\cos n\pi x \, dx$$

$$= \pi\left[(x)\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \pi\left[(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \pi\left[\frac{x\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2}\right]_0^1 + \pi\left[(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - \frac{\cos n\pi x}{n^2\pi^2}\right]_1^2$$

$$= \pi\left[0 + \frac{(-1)^n}{n^2\pi^2} - 0 - \frac{1}{n^2\pi^2}\right] + \pi\left[0 - \frac{1}{n^2\pi^2} - 0 + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \frac{2\pi}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} \frac{-4}{n^2\pi}, & n \ is \ odd \\ 0 \ , & n \ is \ even \end{cases}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x) \ \sin\frac{n\pi x}{l}dx$$

$$= \int_0^2 f(x)\sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x)\sin n\pi x \, dx$$

$= \pi \left[(x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \pi\left[(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$

$= \pi\left[-\frac{x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_0^1 - \pi\left[(2-x)\left(\frac{\cos n\pi x}{n\pi}\right) + \frac{\sin n\pi x}{n^2\pi^2}\right]_1^2$

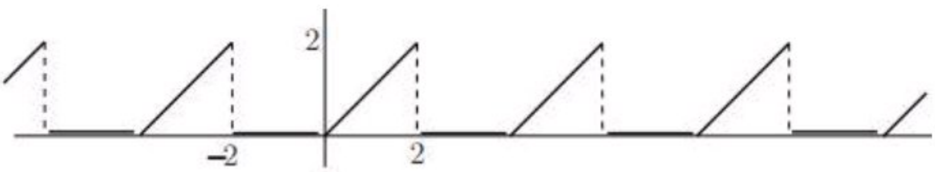
$= \pi\left[-\frac{(-1)^n}{n\pi} + 0 + 0 + 0\right] - \pi\left[0 + 0 - \frac{(-1)^n}{n\pi} + 0\right]$

$= \frac{\pi}{n\pi}[-(-1)^n + (-1)^n] = 0$

Substituting values of a_0 , a_n , b_n in ①

$f(x) \approx \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$

Example 14 Find the Fourier series expansion of the periodic function shown by the graph given below in the interval $(-2,2)$



Solution: From the graph $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$

Clearly $f(x)$ is integrable and piecewise continuous in the interval $(-2,2)$

$\therefore f(x)$ can be expanded into Fourier series given by:

Let $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here interval is $(-2,2)$, $\therefore 2l = 4 \Rightarrow l = 2$

Putting $l = 2$ in ①

$f(x) \approx \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{2} + \sum b_n \sin\frac{n\pi x}{2}$ …… ①

$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} 0\, dx + \frac{1}{2}\int_0^2 x dx = \frac{1}{4}[x^2]_0^2 = 1$

$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{5}dx$

$= \frac{1}{2}\int_{-2}^{0} 0\cos\frac{n\pi x}{l}dx + \frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$

$= 0 + \frac{1}{2}\left[(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$

$= \frac{2}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^2 = \frac{2}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} \frac{-4}{n^2\pi^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$

$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{5}dx$

$= \frac{1}{2}\int_{-2}^{0} 0\sin\frac{n\pi x}{l}dx + \frac{1}{2}\int_0^2 x\sin\frac{n\pi x}{2}dx$

$$= 0 + \frac{1}{2}\left[(x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{-1}{n\pi}\left[x\cos\frac{n\pi x}{2}\right]_0^2 = \frac{-1}{n\pi}[2(-1)^n] = \frac{2(-1)^{n+1}}{n\pi} = \begin{cases} \frac{2}{n\pi}, & n \text{ is odd} \\ \frac{-2}{n\pi}, & n \text{ is even} \end{cases}$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \cdots\right] + \frac{2}{\pi}\left[\frac{\sin\frac{\pi x}{2}}{1} - \frac{\sin\frac{2\pi x}{2}}{2} + \frac{\sin\frac{3\pi x}{2}}{3} - \frac{\sin\frac{4\pi x}{2}}{4} + \cdots\right]$$

$$\Rightarrow f(x) \approx \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right] + \frac{2}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} \ldots\right]$$

1.4 Fourier Series Expansion of Even Odd Functions

Computational procedure of Fourier series can be reduced to great extent, once a function is identified to be even or odd in an interval $(-l, l)$ Characteristics of even/odd functions are given below:

	Even Function	Odd Function
1.	A function $f(x)$ is said to be even if $f(-x) = f(x)$	A function $f(x)$ is said to be odd if $f(-x) = -f(x)$
2.	Graph of an even function is symmetrical about y -axis	Graph of an odd function is symmetrical about origin
3.	For an even function $\int_{-c}^{c} f(x)dx = 2\int_0^c f(x)dx$	For an odd function $\int_{-c}^{c} f(x)dx = 0$
4.	Product of two even functions is even. Product of two odd functions is even	Product of an even and an odd function is odd.
5.	Fourier coefficients of an even function are: $a_0 = \frac{1}{l}\int_{-c}^{c} f(x)dx = \frac{2}{l}\int_0^c f(x)dx$ $a_n = \frac{1}{l}\int_{-c}^{c} f(x)\ \cos\frac{n\pi x}{l}dx$ $= \frac{2}{l}\int_0^c f(x)\ \cos\frac{n\pi x}{l}dx$ as $f(x)$ and $\cos\frac{n\pi x}{l}$ both are even. $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\ \sin\frac{n\pi x}{l}dx = 0$ as $f(x)$ is even and $\sin\frac{n\pi x}{l}$ is odd $\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$	Fourier coefficients of an odd function are: $a_0 = \frac{1}{l}\int_{-c}^{c} f(x)dx = 0$ $a_n = \frac{1}{l}\int_{-c}^{c} f(x)\ \cos\frac{n\pi x}{l}dx = 0$ as $f(x)$ is odd and $\cos\frac{n\pi x}{l}$ is even. $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\ \sin\frac{n\pi x}{l}dx$ $= \frac{2}{l}\int_0^c f(x)\ \sin\frac{n\pi x}{l}dx$ as $f(x)$ and $\sin\frac{n\pi x}{l}$ both are odd $\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Note: Properties of Even or Odd Function comply only if interval is $(-l, l)$ and any function in $(0, 2l)$ does not follow the properties of even/odd functions. For example for the function $f(x) = x^2$ in $(0, 2\pi)$, Fourier coefficients a_0 , a_n , b_n do not follow above given rules of even/odd functions.

Example 15 Obtain Fourier series expansion for the function $f(x) = x^3$ in the interval $(-\pi, \pi)$, if $f(x + 2\pi) = f(x)$

Solution: $f(x) = x^3$ is integrable and piecewise continuous in the interval $(-\pi, \pi)$ and also $f(x)$ is an odd function of x.

$\therefore a_0 = a_n = 0,\ f(x)$ can be expanded into Fourier series given by:

$f(x) \approx \sum_{n=1}^{\infty} b_n sinnx$ ...... ①

$$b_n = \frac{2}{\pi}\int_0^c f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$= \frac{2}{\pi}\left[(x^3)\left(\frac{-\cos nx}{n}\right) - (3x^2)\left(\frac{-\sin nx}{n^2}\right) + (6x)\left(\frac{\cos nx}{n^3}\right) - (6)\left(\frac{\sin nx}{n^4}\right)\right]_o^{\pi}$$

$$= \frac{2}{\pi}\left[(x^3)\left(\frac{-\cos nx}{n}\right) + (6x)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \quad \because sinnx = 0 \text{ when } x = \pi \text{ or } 0$$

$$= \frac{2}{\pi}\left[(\pi^3)\left(\frac{-\cos n\pi}{n}\right) + (6\pi)\left(\frac{\cos n\pi}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[(\pi^3)\left(\frac{-(-1)^n}{n}\right) + (6\pi)\left(\frac{(-1)^n}{n^3}\right)\right]$$

$$= 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore f(x) \approx 2\left[-\left(\frac{-\pi^2}{1} + \frac{6}{1^3}\right)\right] sinx + \left(\frac{-\pi^2}{2} + \frac{6}{2^3}\right) sin2x - \left(\frac{-\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x + \cdots$$

Example 16 If $f(x + 2\pi) = f(x)$, obtain Fourier series expansion for the function given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

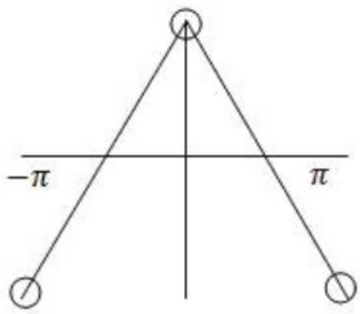
Hence or otherwise prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solution: $f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases} = \begin{cases} 1 - \frac{2x}{\pi}, & \pi \ge x \ge 0 \\ 1 + \frac{2x}{\pi}, & 0 \ge x \ge -\pi \end{cases} = f(x)$

$f(-x) = f(x) \therefore f(x)$ is even function of x

Note: $f(-\pi) = -1, f(0) = 1, f(\pi) = -1$

Plotting graph of $f(x)$, we observe that it is symmetrical about y-axis as shown in given figure. Therefore the function is even.



Hence $b_n = 0, f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$ ...... ①

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = \frac{2}{\pi}[\pi - \pi] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{sinnx}{n}\right)-\left(\frac{-2}{\pi}\right)\left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= -\frac{4}{\pi^2 n^2}[\cos n\pi]_0^{\pi} = \frac{-4}{\pi^2 n^2}[(-1)^n - 1] = \begin{cases}\frac{8}{\pi^2 n^2}, n \text{ is odd}\\ 0, n \text{ is even}\end{cases}$$

Substituting values of a_0 and a_n in ①

$$f(x) \approx \frac{8}{\pi^2}\left[\frac{cosx}{1^2}+\frac{cos3x}{3^2}+\frac{cos5x}{5^2}\right]$$

Putting $x = 0$ on both sides

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

Example 17 If $f(x+2c) = f(x)$, find the Fourier series expansion of the function given by

$$f(x) = \begin{cases}-\sin\frac{\pi x}{c}, & -c \le x \le 0\\ \sin\frac{\pi x}{c}, & 0 \le x \le c\end{cases}$$

Solution: $f(-x) = \begin{cases}\sin\frac{\pi x}{c}, & -c \le -x \le 0\\ -\sin\frac{\pi x}{c}, & 0 \le -x \le c\end{cases} = \begin{cases}\sin\frac{\pi x}{c}, & c \ge x \ge 0\\ -\sin\frac{\pi x}{c}, & 0 \ge x \ge -c\end{cases} = f(x)$

$f(-x) = f(x) \therefore f(x)$ is an even function of $x \Rightarrow b_n = 0$

Also $2l = 2c \Rightarrow l = c$

$$f(x) \approx \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{c} \;\ldots\ldots ①$$

$$a_0 = \frac{2}{c}\int_0^c f(x)dx = \frac{2}{c}\int_0^c \sin\frac{\pi x}{c}dx = -\frac{2}{c}\frac{c}{\pi}\left[\cos\frac{\pi x}{c}\right]_0^c = -\frac{2}{\pi}[-1-1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{c}\int_0^c \sin\frac{\pi x}{c}\, cos\frac{n\pi x}{c}dx$$

$$= \frac{1}{c}\int_0^c\left[\sin\frac{\pi x}{c}(n+1)-\sin\frac{\pi x}{c}(n-1)\right]dx$$

$$= \frac{1}{c}\left[-\frac{c}{\pi(n+1)}\cos\frac{\pi x}{c}(n+1)+\frac{c}{\pi(n-1)}\cos\frac{\pi x}{c}(n-1)\right]_0^c, n \ne 1$$

$$= \frac{1}{\pi}\left[-\frac{1}{(n+1)}\cos(n+1)\pi+\frac{1}{(n-1)}\cos(n-1)\pi+\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)}+\frac{(-1)^{n-1}}{(n-1)}+\frac{1}{(n+1)}-\frac{1}{(n-1)}\right]$$

$$= \begin{cases}\frac{1}{\pi}\left[-\frac{1}{n+1}+\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right], & n \text{ is odd}(n \ne 1)\\ \frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right], & n \text{ is even}\end{cases} = \begin{cases}0, & n \text{ is odd}(n \ne 1)\\ \frac{-4}{\pi(n^2-1)}, & n \text{ is even}\end{cases}$$

$$a_1 = \frac{2}{c}\int_0^c \sin\frac{\pi x}{c}\, cos\frac{\pi x}{c}dx = \frac{1}{c}\int_0^c \sin\frac{2\pi x}{c}dx = \frac{1}{c}\left[-\frac{c}{2\pi}\cos\frac{2\pi x}{c}\right]_0^c$$

$= -\frac{1}{2\pi}[\cos 2\pi - \cos 0] = -\frac{1}{2\pi}[1-1] = 0$

Substituting values of a_0 , a_n in ①

$$f(x) \approx \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{cos\frac{2\pi x}{c}}{2^2-1} + \frac{cos\frac{4\pi x}{c}}{4^2-1} + \frac{cos\frac{6\pi x}{c}}{6^2-1} + \cdots\right]$$

Example 18 If $f(x+2\pi) = f(x)$, obtain Fourier series expansion for the function given by $f(x) = |x|$ in the interval $(-\pi, \pi)$

Hence or otherwise prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solution: $f(-x) = |-x| = |x| = f(x)$

$f(-x) = f(x) \therefore f(x)$ is even function of x.

Rewriting $f(x)$ as $|x| = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$

Being even function of , $b_n = 0$,

$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$ ……①

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}[\pi^2] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(x)\left(\frac{sinnx}{n}\right) - (1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[\cos n\pi]_0^{\pi} = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} \frac{-4}{\pi n^2}, n\ is\ odd \\ 0, n\ is\ even \end{cases}$$

Substituting values of a_0 and a_n in ①

$$f(x) \approx \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{cosx}{1^2} + \frac{cos3x}{3^2} + \frac{cos5x}{5^2}\right]$$

Putting $x = 0$ on both sides

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 19 Obtain Fourier series expansion for the function given by $f(x) = |\sin x|$ in the interval $(-\pi, \pi)$

Solution: $f(x) = |\sin x|$ is even function of x

$\therefore b_n = 0, \quad f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$ ……①

Rewriting $f(x)$ as $|\sin x| = \begin{cases} -\sin x, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{-2}{\pi}\left[\cos x\right]_0^\pi = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi \sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi (\sin(n+1)\,x - sin(n-1)xdx)$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], n\ is\ odd(n \neq 1) \\ \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right], \quad n\ is\ even \end{cases} = \begin{cases} 0, \quad n\ is\ odd(n \neq 1) \\ \frac{-4}{\pi(n^2-1)}, \quad n\ is\ even \end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx$$

$$= \frac{-1}{2\pi}[\cos 2x]_0^\pi = -\frac{1}{2\pi}[\cos 2\pi - \cos 0] = -\frac{1}{2\pi}[1-1] = 0$$

Substituting values of a_0 , a_n in ①

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{cos2x}{2^2-1} + \frac{cos4x}{4^2-1} + \frac{cos6x}{6^2-1} + \cdots\right]$$

$$\Rightarrow f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}cos2x + \frac{1}{15}cos4x + \frac{1}{35}cos6x + \cdots\right]$$

Example 20 Obtain Fourier series expansion for the function given by $f(x) = |\cos x|$ in the interval $(-\pi, \pi)$

Solution: $f(x) = |\cos x|$ is even function of x

$$\therefore b_n = 0, \quad f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx \quad \ldots\ldots ①$$

Rewriting $f(x)$ as $|\cos x| = \begin{cases} -cosx, & -\pi < x < -\frac{\pi}{2} \\ cosx, & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ -cosx, & \frac{\pi}{2} \le x < \pi \end{cases}$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} -\cos x\,dx = \frac{2}{\pi}[\sin x]_0^{\frac{\pi}{2}} - \frac{2}{\pi}[\sin x]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\sin\frac{\pi}{2} - \sin 0 - \sin\pi + \sin\frac{\pi}{2}\right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cos nx\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} -\cos x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (\cos(n+1)\,x + cos(n-1)xdx) - \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\cos(n+1)\,x + cos(n-1)xdx)$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} + \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right] \;, (n \neq 1)$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right] \;, (n \neq 1)$$

$$= \begin{cases} 0 & , n \text{ is odd } \; (n \neq 1) \\ \frac{2}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] , & n = 2,6,10\ldots \\ \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] , & n = 4,8,12 \ldots \end{cases} = \begin{cases} 0, & n \text{ is odd}(n \neq 1) \\ \frac{4}{\pi(n^2-1)} , & n = 2,6,10 \ldots \\ \frac{-4}{\pi(n^2-1)} , & n = 4,8,12\ldots \end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x \, dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x \cos x \, dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} -\cos x \cos x \, dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^2 x \, dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos^2 x \, dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (1 + \cos 2x \, dx - \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} (1 + \cos 2x)\, dx$$

$$= \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - 0 - 0 - \pi - \frac{\sin 2\pi}{2} + \frac{\pi}{2} + \frac{\sin\pi}{2}\right] = 0$$

Substituting values of a_0 , a_n in ①

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{cos2x}{2^2-1} - \frac{cos4x}{4^2-1} + \frac{cos6x}{6^2-1} - \cdots\right]$$

$$\Rightarrow f(x) = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{1}{3}cos2x - \frac{1}{15}cos4x + \frac{1}{35}cos6x + \cdots\right]$$

Exercise 1A

1. If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \frac{\pi x}{4}, & 0 < x < \pi \end{cases}$$

   Hence or otherwise prove that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

2. If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of the function

$$f(x) = \begin{cases} -\pi - x, & \pi \leq x \leq 0 \\ \pi + x, & 0 \leq x \leq \pi \end{cases}$$

   Hence or otherwise prove that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

3. Find the Fourier series expansion of the function $f(x) = x - x^2$ in $(-\pi, \pi)$.

   Hence deduce that $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

4. Find the Fourier series expansion of the function $f(x) = \frac{(\pi - x)^2}{4}$ in $(0, 2\pi)$.
5. If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ \frac{1}{2}, & x = 0 \\ x, & 0 < x < 1 \end{cases}$$

6. Assuming $f(x) = x$ to be periodic with a period 2π, find Fourier series expansion of $f(x)$ in the interval $(-\pi, \pi)$.
7. If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of the function

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

8. If $f(x + 1) = f(x)$, find Fourier series expansion of $f(x) = x$ in the interval $(0,1)$.
9. If $f(x + 4) = f(x)$, find Fourier series expansion of $f(x) = |x|$ in the interval $(-2,2)$.
10. If $f(x + 2) = f(x)$, find the Fourier series expansion of the function

$$f(x) = \begin{cases} -2, & -1 < x < 0 \\ 2, & 0 < x < 1 \end{cases}$$

11. If $f(x + 2\pi) = f(x)$, find the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi + x, & -\pi \le x \le -\frac{\pi}{2} \\ \frac{\pi}{2}, & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

12. find the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi x, & 0 \le x < 1 \\ 0, & x = 1 \\ \pi(x - 2), & 1 < x \le 2 \end{cases}$$

Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Answers

1. $f(x) \approx \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \frac{\pi}{4}\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$

2. $f(x) \approx \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + 4\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$

3. $f(x) \approx \frac{-\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$

4. $f(x) \approx \frac{\pi^2}{12} + \left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right]$

5. $f(x) \approx \frac{3}{4} - \frac{2}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right] - \frac{1}{\pi}\left[\frac{\sin \pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \cdots\right]$

6. $f(x) \approx 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$

7. $f(x) \approx \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

8. $f(x) \approx \frac{1}{2} - \frac{1}{\pi}\left[\frac{\sin 2\pi x}{1} + \frac{\sin 4\pi x}{2} + \frac{\sin 6\pi x}{3} + \cdots\right]$

9. $f(x) \approx 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \cdots\right]$

10. $f(x) \approx \frac{8}{\pi}\left[\frac{\sin \pi x}{1} + \frac{\sin 3\pi x}{3} + \frac{\sin 5\pi x}{5} + \cdots\right]$

11. $f(x) \approx \frac{3\pi}{8} + \frac{1}{\pi}\left[\frac{2\cos x}{1^2} - \frac{\cos 2x}{1^2} + \frac{2\cos 3x}{3^2} - \frac{\cos 6x}{3^2} + \frac{2\cos 5x}{5^2} - \frac{\cos 10x}{5^2} + \cdots\right]$

12. $f(x) \approx 2\left[\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \cdots\right]$

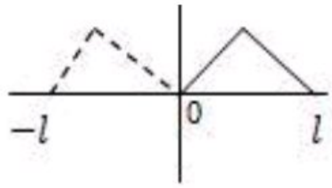
1.6 Half Range Fourier Series in the Interval $(0, l)$

Sometimes it is required to expand a function $f(x)$ in the range $(0, \pi)$ or more generally in $(0, l)$ into a sine series or cosine series.

If it is required to expand $f(x)$ in $(0, l)$, it is immaterial what the function may be outside the range $0 < x < l$, we are free to choose the function in $(-l, 0)$.

1.6.1 Half Range Cosine Series

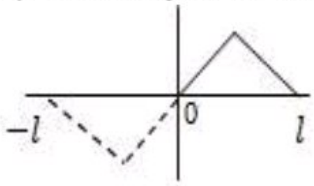
To develop into Cosine series, we extend $f(x)$ in $(-l, 0)$ by reflecting it in y – axis as shown in adjoining figure, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$



$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

1.6.2 Half Range Sine Series

To develop into Sine series, we extend $f(x)$ in $(-l, 0)$, by reflecting it in origin, so that $f(-x) = -f(x)$, function becomes odd function and $a_0 = a_n = 0$



$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Example 21 Obtain half range Fourier Cosine series for $f(x) = 2x - 1$ in the interval (0,1).

Solution: To develop $f(x) = 2x - 1$ into Cosine series, extending $f(x)$ in $(-1, 0)$ by reflecting it in y – axis, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$

$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ …… ①

Here $2l = 2 \quad \therefore l = 1$

$a_0 = \frac{2}{l}\int_0^l f(x)dx = 2\int_0^1 (2x-1)dx = 0$

$a_n = 2\int_0^1 (2x-1)\cos n\pi x\, dx$

$$= 2\left[(2x-1)\left(\frac{\sin n\pi x}{n\pi}\right) - (2)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}[\cos n\pi - \cos 0]$$

$$= \frac{4}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} \frac{-8}{n^2\pi^2} & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases}$$

Substituting values of a_0 , a_n in ①

$$f(x) \approx -\frac{8}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$$

Example 22 Obtain half range Fourier Cosine series for $f(x) = x \sin x$ in the interval $(0, \pi)$ and show that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi - 2}{4}$

Solution: To develop $f(x) = x \sin x$ into Cosine series, extending $f(x)$ in $(-\pi, 0)$ by reflecting it in y – axis, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \;\ldots\ldots ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, sinx dx = \frac{2}{\pi}\,[(x)(-cosx) - (1)(-sinx)]_0^{\pi}$$

$$= \frac{-2}{\pi}[xcosx]_0^{\pi} = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \,.\, \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[\sin(n+1)\,x - \sin\,(n-1)x]dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos\,(n+1)x}{(n+1)} + \frac{\cos\,(n-1)x}{(n-1)}\right) - (1)\left(\frac{-\sin\,(n+1)}{(n+1)^2} + \frac{\sin\,(n-1)}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos\,(n+1)x}{(n+1)} + \frac{x\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{\pi}{\pi}\left[-\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}\right]$$

$$= (-1)^{n+1}\left[\frac{n+1-n+1}{(n-1)(n+1)}\right] \qquad \because (-1)^{n-1} = (-1)^{n+1}$$

$$= \frac{2(-1)^{n+1}}{(n-1)(n+1)}, \quad n \neq 1$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{-1}{2\pi}[x\cos 2x]_0^{\pi} = \frac{-1}{2\pi}[\pi] = -\frac{1}{2}$$

$$\therefore f(x) = 1 - \frac{1}{2}cosx + 2\left[-\frac{\cos 2x}{1.3} + \frac{\cos 3x}{2.4} - \frac{\cos 4x}{3.5} + \frac{\cos 5x}{4.6} - \frac{\cos 6x}{5.7} + \cdots\right]$$

Putting $x = \frac{\pi}{2}$ on both sides

$$\frac{\pi}{2} = 1 + 2\left[\frac{1}{1.3} + 0 - \frac{1}{3.5} + 0 + \frac{1}{5.7} + \cdots\right]$$

$$\Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi-2}{4}$$

Example 23 Obtain half range Fourier Sine and Cosine series for the function given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

Solution: <u>Fourier Sine series</u>

To develop $f(x)$ into Sine series, extending $f(x)$ in $(-\pi, 0)$ by reflecting in origin, so that $f(-x) = -f(x)$, function becomes odd function and $a_0 = a_n = 0$

$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin nx$ …… ①

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x \sin nx\, dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[-(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2} = \begin{cases} 0, & when\ n\ is\ even \\ \frac{4}{\pi n^2}, & n = 1, 5, 9, \ldots \\ \frac{-4}{\pi n^2}, & n = 3, 7, 11, \ldots \end{cases}$$

Substituting b_n in ①

$$f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \frac{1}{49}\sin 7x \ldots\right]$$

<u>Fourier Cosine series</u>

To develop $f(x)$ into Cosine series, extending $f(x)$ in $(-\pi, 0)$ by reflecting it in $y-$axis, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$

$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ …… ②

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)dx\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{1}{n^2}\cos n\pi - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$= \frac{2}{\pi n^2}\begin{cases} 0, & n \text{ is odd} \\ 2(-1) - 1 - 1, & n = 2,6,10,\ldots \\ 2(1) - 1 - 1, & n = 4,8,12,\ldots \end{cases} = \begin{cases} 0, & n \text{ is odd} \\ \frac{-8}{\pi n^2}, & n = 2,6,10,\ldots \\ 0, & n = 4,8,12,\ldots \end{cases}$$

Substituting a_0, a_n in ②

$$f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{\cos 2x}{4} + \frac{\cos 6x}{36} + \frac{\cos 10x}{100} + \cdots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1} + \frac{\cos 6x}{9} + \frac{\cos 10x}{25} + \cdots\right]$$

Example 24 Obtain half range Fourier Sine series for $f(x) = \begin{cases} \frac{2kx}{l}, & 0 \le x \le \frac{l}{2} \\ \frac{2k}{l}(l - x), & \frac{l}{2} \le x \le l \end{cases}$

Solution: To develop $f(x)$ into Sine series, extending $f(x)$ in $(-l, 0)$ by reflecting in origin, so that $f(-x) = -f(x)$, function becomes odd function and $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \ldots\ldots ①$$

$$b_n = \frac{2}{l}\int_0^l f(x)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^{\frac{l}{2}} \frac{2kx}{l}\,\sin\frac{n\pi x}{l}\,dx + \frac{2}{l}\int_{\frac{l}{2}}^{l} \frac{2k}{l}(l - x)\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4k}{l^2}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{2}} +$$

$$\frac{4k}{l^2}\left[(l - x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{\frac{l}{2}}^{l}$$

$$= \frac{4k}{l^2}\left[-\frac{lx}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{\frac{l}{2}} + \frac{4k}{l^2}\left[-(l - x)\frac{l}{n\pi}\cos\frac{n\pi x}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{\frac{l}{2}}^{l}$$

$$= \frac{4k}{l^2}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8k}{n^2\pi^2}\left[\sin\frac{n\pi}{2}\right] = \begin{cases} \frac{8k}{n^2\pi^2}\,, & n = 1, 5, 9, \dots \\ \frac{-8k}{n^2\pi^2}\,, & n = 3, 7, 11, \dots \\ 0\,, & n\ is\ even \end{cases}$$

Substituting values of b_n in ①

$$f(x) \approx \frac{8k}{n^2\pi^2}\left[\sin\frac{\pi x}{l} - \frac{1}{9}\sin\frac{3\pi x}{l} + \frac{1}{25}\sin\frac{5\pi x}{l} - \frac{1}{49}\sin\frac{7\pi x}{l} + \cdots\right]$$

1.7 Practical Harmonic Analysis

In many engineering and scientific problems, $f(x)$ is not given directly, rather set of discrete values of function are given in the form $(x_i, y_i), i = 1, 2,3, \dots, m$ where x_i's are equispaced. The process of obtaining $f(x)$ in terms of Fourier series from given set of values (x_i, y_i), is known as practical harmonic analysis.

In a given interval $(0,2l)$, $f(x)$ is represented in terms of harmonics as shown below:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

Where $n = 1,2,3$ give 1st, 2nd and 3rd harmonics respectively.

$\therefore \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right)$ is the first harmonic

$\left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right)$ is the second harmonic

$\left(a_3 \cos\frac{\pi x}{l} + b_3 \sin\frac{\pi x}{l}\right)$ is the third harmonic

⋮

$\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$ is the n^{th} harmonic

Fourier coefficient a_0 is computed using the relation

2 [Mean value of y in the interval $(0,2l)$]

$\therefore a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i$, where m denotes number of observations

Similarly a_n and b_n can be found out using the relations

$$a_n = 2\ [\text{Mean value of } y\cos\frac{n\pi x}{l} \text{ in the interval } (0,2l)\ ] = \frac{2}{m}\sum_{i=1}^{m} y_i \cos\frac{n\pi x_i}{l}$$

$$b_n = 2\ \left[\text{Mean value of } y\sin\frac{n\pi x}{l} \text{ in the interval } (0,2l)\right] = \frac{2}{m}\sum_{i=1}^{m} y_i \sin\frac{n\pi x_i}{l}$$

Also when interval length is 2π, putting $2l = 2\pi$ i.e. $l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i\,, \qquad a_n = \frac{2}{m}\sum_{i=1}^{m} y_i \cos nx_i\,, \qquad b_n = \frac{2}{m}\sum_{i=1}^{m} y_i \sin nx_i$$

- The amplitude of first harmonic is given by $\sqrt{a_1^2 + b_1^2}$ and similarly amplitudes of second and third harmonics are given by $\sqrt{a_2^2 + b_2^2}$ and $\sqrt{a_3^2 + b_3^2}$ respectively.
- For $f(x)$ in discrete form, values of Fourier coefficients a_0, a_n and b_n have been computed using trapezoidal rule for definite integration.

Example 25 The following values of 'y' give the displacement of a machine part for the rotation x of a flywheel. Express 'y' in Fourier series up to third harmonic.

x	0^0	60^0	120^0	180^0	240^0	300^0	360^0
y	1.98	2.15	2.77	-0.22	-0.31	1.43	1.98

Solution: Here number of observations (m) are 6, period length is $2\pi \because [y]_{0^0} \equiv [y]_{360^0}$

Let $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$\therefore y \approx \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x)$ ... ①

$$a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i\ ,\ a_n = \frac{2}{m}\sum_{i=1}^{m} y_i \cos nx_i\ ,\ b_n = \frac{2}{m}\sum_{i=1}^{m} y_i \sin nx_i$$

x_i	y_i	$\cos x_i$	$\sin x_i$	$\cos 2x_i$	$\sin 2x_i$	$\cos 3x_i$	$\sin 3x_i$
0^0	19.8	1.0	0	1.0	0	1.0	0
60^0	2.15	0.5	0.866	-0.5	0.866	-1.0	0
120^0	2.77	-0.5	0.866	-0.5	-0.866	1.0	0
180^0	-0.22	-1	0	1.0	0	-1.0	0
240^0	-0.31	-0.5	-0.866	-0.5	0.866	1.0	0
300^0	1.43	0.5	-0.866	-0.5	-0.866	-1.0	0

$a_0 = \frac{2}{6}\sum_{i=1}^{6} y_i = \frac{2}{6}[1.98 + 2.15 + 2.77 - 0.22 - 0.31 + 1.4] = 2.6$

$a_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos x_i = \frac{2}{6}[(1.98)(1) + (2.15)(0.5) + \cdots + (1.43)(0.5)] = 0.92$

$b_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin x_i = \frac{2}{6}[(1.98)(0) + (2.15)(0.866) + \cdots + (1.43)(-0.866)] = 1.097$

$a_2 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos 2x_i = \frac{2}{6}[(1.98)(1) + (2.15)(-0.5) + \cdots + (1.43)(-0.5)] = -0.42$

$b_2 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin 2x_i = \frac{2}{6}[(1.98)(0) + (2.15)(0.866) + \cdots + (1.43)(-0.866)] = -0.681$

$a_3 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos 3x_i = \frac{2}{6}[(1.98)(1) + (2.15)(-1) + \cdots + (1.43)(-1)] = 0.36$

$b_3 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin 3x_i = \frac{2}{6}[(1.98)(0) + (2.15)(0) + \cdots + (1.43)(0)] = 0$

Substituting values of a_0 , a_n , b_n in ① where $n = 1,2,3$

$y \approx 1.3 + (0.92 \cos x + 1.097 \sin x) - (0.42 \cos 2x + 0.681 \sin 2x) + 0.36 \cos 3x + \cdots$

Example 26 Experimental values of y corresponding to x are tabulated below:

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
y	298	356	373	337	254	155	80	51	60	93	147	221	298

Express y in Fourier series up to second harmonic.

Solution: Here number of observations (m) are 12, period length is $2\pi \because [y]_0 \equiv [y]_{2\pi}$

Let $y \approx \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$ ①

$a_0 = \frac{2}{m} \sum_{i=1}^{m} y_i$, $a_n = \frac{2}{m} \sum_{i=1}^{m} y_i \cos nx_i$, $b_n = \frac{2}{m} \sum_{i=1}^{m} y_i \sin nx_i$

x_i	y_i	$\cos x_i$	$\sin x_i$	$\cos 2x_i$	$\sin 2x_i$
0	298	1	0	1	0
$\frac{\pi}{6}$	356	0.866	0.5	0.5	0.866
$\frac{2\pi}{6}$	373	0.5	0.866	-0.5	0.866
$\frac{3\pi}{6}$	337	0	1	-1	0
$\frac{4\pi}{6}$	254	-0.5	0.866	-0.5	-0.866
$\frac{5\pi}{6}$	155	-0.866	0.5	0.5	-0.866
π	80	-1	0	1	1
$\frac{7\pi}{6}$	51	-0.866	-0.5	0.5	0.866
$\frac{8\pi}{6}$	60	-0.5	-0.866	-0.5	0.866
$\frac{9\pi}{6}$	93	0	-1	-1	0
$\frac{10\pi}{6}$	147	0.5	-0.866	-0.5	-0.866
$\frac{11\pi}{6}$	221	0.866	-0.5	0.5	-0.866

$a_0 = \frac{2}{12} \sum_{i=1}^{12} y_i = \frac{1}{6}[298 + 356 + \cdots + 221] = 404.17$

$a_1 = \frac{2}{12} \sum_{i=1}^{12} y_i \cos x_i = \frac{1}{6}[(298)(1) + (356)(0.866) + \cdots + (221)(0.866)] = 107.048$

$b_1 = \frac{2}{12} \sum_{i=1}^{12} y_i \sin x_i = \frac{1}{6}[(298)(0) + (356)(0.5) + \cdots + (221)(-0.5)] = 121.203$

$a_2 = \frac{2}{12} \sum_{i=1}^{12} y_i \cos 2x_i = \frac{1}{6}[(298)(1) + (356)(0.5) + \cdots + (221)(0.5)] = -13$

$b_2 = \frac{2}{12} \sum_{i=1}^{12} y_i \sin 2x_i = \frac{1}{6}[(298)(0) + (356)(0.866) + \cdots + (221)(-0.866)] = 9.093$

Substituting values of a_0 , a_1 , b_1, a_2 , b_2 in ①

$y \approx 202.09 + (107.048 \cos x + 121.203 \sin x) + (-13 \cos 2x + 9.093 \sin 2x) + \cdots$

Example 27 The following table connects values of x and y for a statistical input:

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Express y in Fourier series up to first harmonic. Also find amplitude of the first harmonic.

Solution: Here $m = 6$, Also putting $2l = 6 \Rightarrow l = 3 \because [y]_{x=0} \equiv [y]_{x=6}$, if y is periodic

$$\therefore y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{3} + b_n \sin\frac{n\pi x}{3}\right)$$

$$\Rightarrow y \approx \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \cdots \qquad \ldots ①$$

$$a_0 = \frac{2}{6}\sum_{i=1}^{6} y_i\ ,\ a_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos\frac{\pi x_i}{3},\ b_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin\frac{\pi x_i}{3}$$

x_i	y_i	$\cos\frac{\pi x_i}{3}$	$\sin\frac{\pi x_i}{3}$
0	9	1	0
1	18	0.5	0.866
2	24	-0.5	0.866
3	28	-1	0
4	26	-0.5	-0.866
5	20	0.5	-0.866

$$a_0 = \frac{2}{6}\sum_{i=1}^{6} y_i = \frac{1}{3}[9 + 18 + 24 + 28 + 26 + 20] = 41.67$$

$$a_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos\frac{\pi x_i}{3} = \frac{1}{3}[(9)(1) + (18)(0.5) + \cdots + (20)(0.5)] = -8.33$$

$$b_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin\frac{\pi x_i}{3} = \frac{1}{3}[(9)(0) + (18)(0.866) + \cdots + (20)(-0.866)] = -1.15$$

Substituting values of a_0 , a_1 , b_1 in ①

$$\Rightarrow y \approx 20.835 - \left(8.33 \cos\frac{\pi x}{3} + 1.15 \sin\frac{\pi x}{3}\right) + \cdots$$

The amplitude of first harmonic is given by $\sqrt{(-8.33)^2 + (-1.15)^2} = 8.41$

Example 28 The following table gives the variation of a periodic current over a period 'T'

Time(t) Sec	0	T/6	T/3	T/2	2T/3	5T/6	T
Current(A) Amp	1.98	1.30	1.05	1.3	-0.88	-0.25	1.98

Show that there is a direct current part of 0.75 amp in the variable current. Also obtain the amplitude of the first harmonic.

Solution: Let $A \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{l}$

Here $m = 6$, Also $2l = T \Rightarrow l = \frac{T}{2}$

$$\therefore A \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}$$

$$\Rightarrow A \approx \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T} \qquad \text{for the first harmonic} \ldots ①$$

$$a_0 = \frac{2}{m}\sum A,\ a_1 = \frac{2}{m}\sum A \cos\frac{2\pi t}{T},\ b_1 = \frac{2}{m}\sum A \sin\frac{2\pi t}{T}$$

Time(t) sec	Current(A) amp	$\cos\frac{2\pi t}{T}$	$\sin\frac{2\pi t}{T}$
0	1.98	1	0
T/6	1.3	0.5	0.866
T/3	1.05	-0.5	0.866
T/2	1.3	-1	0
2T/3	-0.88	-0.5	-0.866
5T/6	-0.25	0.5	-0.866

$a_0 = \frac{2}{6}\sum A = \frac{1}{3}[1.98 + 1.3 + 1.05 + 1.3 - 0.88 - 0.25] = 1.5$

$a_1 = \frac{2}{6}\sum A\cos\frac{2\pi t}{T} = \frac{1}{3}[(1.98)(1) + (1.3)(0.5) + \cdots + (-0.25)(0.5)] = 0.373$

$b_1 = \frac{2}{6}\sum A\sin\frac{2\pi t}{T} = \frac{1}{3}[(1.98)(0) + (1.3)(0.866) + \cdots + (-0.25)(-0.866)] = 1.005$

Substituting values of a_0 , a_1 , b_1 in ①

$\therefore A \approx 0.75 + 0.373\cos\frac{2\pi t}{T} + 1.005\sin\frac{2\pi t}{T}$

Here $\frac{a_0}{2}$ represents the direct current part and the amplitude of the first harmonic is given by $\sqrt{a_1^2 + b_1^2}$

$\therefore A$ has a direct current part of 0.75 amp

The amplitude of first harmonic is given by $\sqrt{(0.373)^2 + (1.005)^2} = \sqrt{1.1491} = 1.072$

1.7.1 Harmonic Analysis for Half Range Series

If it is required to express $f(x)$ given in discrete form $(x_i, y_i), i = 1, 2,3, \dots, m$, taken in the interval $(0, l)$ into half range sine or cosine series, we extend $f(x)$ in $(-l, 0)$ to make it odd or even respectively.

Sine Series

To develop $f(x)$ into sine series, extend $f(x)$ in the interval $(-l, 0)$ by reflecting in origin, so that $f(-x) = -f(x)$, function becomes odd function and $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = 2\left[\text{Mean value of } y\sin\frac{n\pi x}{l} \text{ in the interval } (0, l)\right] = \frac{2}{m}\sum_{i=1}^{m} y_i \sin\frac{n\pi x_i}{l}$$

Note: To express $f(x)$ into sine series, y_1 must be zero, otherwise it cannot be reflected in origin.

Cosine Series

To develop $f(x)$ into cosine series, extend $f(x)$ in the interval $(-l, 0)$ by reflecting in y-axis, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$a_0 = 2$ [Mean value of y in the interval $(0, l)$]

$$= \frac{2}{m}\left[\frac{y_1+y_m}{2} + y_2 + y_3 + \cdots + y_{m-1}\right] \quad \ldots\ldots \text{ using trapezoidal rule}$$

$$a_n = 2\left[\text{Mean value of } y\cos\frac{n\pi x}{l} \text{ in the interval } (0, l)\right]$$

$$= \frac{2}{m}\left[\frac{y_1\cos\frac{n\pi x}{l} + y_m\cos\frac{n\pi x}{l}}{2} + y_2\cos\frac{n\pi x}{l} + y_3\cos\frac{n\pi x}{l} + \cdots + y_{m-1}\cos\frac{n\pi x}{l}\right]$$

$$n = 1,2,3 \ldots$$

Example 29 The turning moment 'M' units of a crank shaft of a steam engine are given for a series of values of the crank angle 'θ' in degrees. Obtain first three terms of sine series to represent M. Also verify the value M from obtained function at $\theta = 60^0$

θ	0^0	30^0	60^0	90^0	120^0	150^0
M	0	5224	8097	7850	5499	2656

Solution: Assuming M periodic, to represent into sine series (half range series), extending M in the interval $(-180^0, 0)$ by reflecting in origin, so that $M(-\theta) = -M(\theta)$, function becomes odd function and $a_0 = a_n = 0$

$$\therefore M \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi\theta}{l}$$

Here $m = 6$, Also $2l = 2\pi \Rightarrow l = \pi$

$$\Rightarrow M \approx \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$\Rightarrow \text{M} \approx b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + \cdots$$

$$b_n = \frac{2}{6}\sum M\sin n\theta\ ,\ n = 1,2,3 \ldots$$

θ	M	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$
0^0	0	0	0	0
30^0	5224	0.5	0.866	1
60^0	8097	0.866	0.866	0
90^0	7850	1	0	-1
120^0	5499	0.866	-0.866	0
150^0	2656	0.5	-0.866	1

$$b_1 = \frac{2}{6}\sum M\sin\theta = \frac{1}{3}[(0)(0) + (5224)(0.5) + \cdots + (2656)(0.5)] = 7850$$

$$b_2 = \frac{2}{6}\sum M\sin 2\theta = \frac{1}{3}[(0)(0) + (5224)(0.866) + \cdots + (2656)(-0.866)] = 1500$$

$$b_3 = \frac{2}{6}\sum M\sin 3\theta = \frac{1}{3}[(0)(0) + (5224)(1) + \cdots + (2656)(1)] = 0$$

$$\therefore \text{M} \approx 7850\sin\theta + 1500\sin 2\theta + 0 + \cdots$$

When $\theta = 60^0$, $\text{M} \approx 7850\sin 60^0 + 1500\sin 120^0 + 0 + \cdots$

$$\approx 7850(0.866) + 1500(0.866) + \cdots$$

$$\approx 8097.1$$

Example 30 Obtain half range Fourier cosine series for the data given below:

x	0	1	2	3	4	5
y	4	8	11	15	12	7

Also check value of y at $x = 2$ from the obtained cosine series.

Solution: Assuming y periodic, to represent it into half range cosine series, extending y in the interval $(-6,0)$ by reflecting it in y-axis, so that $y(-x) = y(x)$, function becomes even function and $b_n = 0$

$$\therefore y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Here $m = 6$, Also $2l = 12 \Rightarrow l = 6$

$$\Rightarrow y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{6}$$

$$\Rightarrow y \approx \frac{a_0}{2} + a_1 \cos\frac{\pi x}{6} + a_2 \cos\frac{2\pi x}{6} + a_3 \cos\frac{3\pi x}{6} + \cdots$$

$$a_0 = \frac{2}{m}\left[\frac{y_1+y_m}{2} + y_2 + y_3 + \cdots + y_{m-1}\right]$$

$$a_n = \frac{2}{m}\left[\frac{y_1\cos\frac{n\pi x}{l} + y_m\cos\frac{n\pi x}{l}}{2} + y_2\cos\frac{n\pi x}{l} + y_3\cos\frac{n\pi x}{l} + \cdots + y_{m-1}\cos\frac{n\pi x}{l}\right]$$

$n = 1,2,3 \ldots$

x_i	y_i	$\cos\frac{\pi x}{6}$	$\cos\frac{2\pi x}{6}$	$\cos\frac{3\pi x}{6}$	$y\cos\frac{\pi x}{6}$	$y\cos\frac{2\pi x}{6}$	$y\cos\frac{3\pi x}{6}$
0	4	1	1	1	4	4	4
1	8	0.866	0.5	0	0.6928	4	0
2	11	0.5	-0.5	-1	5.5	-5.5	-11
3	15	0	-1	0	0	-15	0
4	12	-0.5	-0.5	1	-6	-6	12
5	7	-0.866	0.5	0	-6.062	3.5	0

$$a_0 = \frac{2}{6}\left[\frac{4+7}{2} + 8 + 11 + 15 + 12\right] = \frac{1}{3}[51.5] = 17.2$$

$$a_1 = \frac{2}{6}\left[\frac{4-6.062}{2} + 0.6928 + 5.5 + 0 - 6\right] = \frac{1}{3}[-0.8382] = -0.2794$$

$$a_2 = \frac{2}{6}\left[\frac{4+3.5}{2} + 4 - 5.5 - 15 - 6\right] = \frac{1}{3}[-18.75] = -6.25$$

$$a_3 = \frac{2}{6}\left[\frac{4+0}{2} + 0 - 11 + 0 + 12\right] = \frac{1}{3}[3] = 1$$

$$\therefore y \approx 8.6 - 0.2794\cos\frac{\pi x}{6} - 6.25\cos\frac{2\pi x}{6} + \cos\frac{3\pi x}{6} + \cdots$$

When $x = 2$, $y \approx 8.6 - 0.2794\cos\frac{2\pi}{6} - 6.25\cos\frac{4\pi}{6} + \cos\frac{6\pi}{6} + \cdots$

$$\approx 8.6 - 0.2794(0.5) - 6.25(-0.5) - 1 + \cdots$$

$$\approx 10.5853$$

Exercise 1B

1. Find half range Fourier cosine series expansion of the function

$$f(x) = (\pi - x) \text{ in the interval } (0, \pi)$$

2. Find half range Fourier cosine series expansion of the function

$$f(x) = \sin\frac{\pi x}{l} \text{ in the interval } (0, 1)$$

3. Find half range Fourier sine series expansion of the function

$$f(x) = e^x \text{ in the interval } (0, 1)$$

4. Find the Fourier sine series expansion of the function

$$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \end{cases}$$

5. The following values of 'y' give the displacement of a machine part for the rotation x of a flywheel. Express 'y' in Fourier series.

x	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0
y	70	886	1293	1400	1307	814	-70	-886	-1293	-1400	-1307	-814

Express y in Fourier series up to third harmonics.

6. y is a discrete function of x, given as below:

x	0^0	60^0	120^0	180^0	240^0	300^0	360^0
y	0.8	0.6	0.4	0.7	0.9	1.1	0.8

Express y in Fourier series up to first harmonic.

7. Obtain Fourier sine series of y for the data given below:

x	0	1	2	3	4	5	6
y	0	10	15	8	5	3	0

8. Following values of y give the displacement of a certain machine part for the movement x of another part:

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
y	0	9.2	14.4	17.8	17.3	11.7	0

Express y in Fourier series up to second harmonic.